

Measure Theory

Classes of sets

Definition 1.1 (σ -algebra): A class of sets $A \subset 2\Omega$ if it fulfils the following three conditions: (i) $\Omega \in A$. (ii) A is closed under complements. (iii) A is closed under countable unions.

Definition 1.2 (algebra): A class of sets $A \subset 2\Omega$ is called an algebra if the following three conditions are fulfilled: (i) $\Omega \in A$. (ii) A is $\setminus$ -closed. (iii) A is $\cup$ -closed

Definition 1.3 (ring): A class of sets $A \subset 2\Omega$ is called a ring if the following three conditions hold: (i) $\emptyset \in A$. (ii) A is $\setminus$ -closed. (iii) A is $\cup$ -closed. A ring is called a σ -ring if it is also σ - $\cup$ -closed

Definition 1.4 (semiring): A class of sets $A \subset 2\Omega$ is called a semiring if (i) $\emptyset \in A$, (ii) for any two sets $A, B \in A$ the difference set $B \setminus A$ is a finite union of mutually disjoint sets in A , (iii) A is $\cap$ -closed.

Definition 1.5 (Dynkin-system): A class of sets $A \subset 2\Omega$ is called a λ -system (or Dynkin's λ -system) if (i) $\Omega \in A$, (ii) for any two sets $A, B \in A$ with $A \subset B$, the difference set $B \setminus A$ is in A , and (iii) $\cup_{n=1}^{\infty} A_n \in A$ for any choice of countably many pairwise disjoint sets $A_1, A_2, \dots \in A$.

Definition 1.6 (liminf & limsup): Let $A_1, A_2, \dots$ be subsets of Ω . The sets $\liminf_{n \rightarrow \infty} A_n := \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m$ and $\limsup_{n \rightarrow \infty} A_n := \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$ are called limes inferior and limes superior, respectively, of the sequence $(A_n)_{n \in \mathbb{N}}$.

Theorem 1.1 (Intersection of classes of sets): Let I be an arbitrary index set, and assume that A_i is a σ -algebra for every $i \in I$. Hence the intersection $A_I := \{A \subset \Omega : A \in A_i \text{ for every } i \in I\} = \cap_{i \in I} A_i$ is a σ -algebra. The analogous statement holds for rings, σ -rings, algebras and λ -systems. However, it fails for semirings

Theorem 1.2 (Generated σ -algebra): Let $E \subset 2\Omega$. Then there exists a smallest σ -algebra $\sigma(E)$ with $E \subset \sigma(E)$: $\sigma(E) := \cap \{A \subset 2\Omega : A \text{ is a } \sigma\text{-algebra } A \supset E\}$.

Theorem 1.3 (π -closed λ -system): Let $D \subset 2\Omega$ be a λ -system. Then D is a π -system $\Leftrightarrow D$ is a σ -algebra.

Theorem 1.4 (Dynkin's π - λ theorem): If $E \subset 2\Omega$ is a π -system, then $\sigma(E) = \delta(E)$.

Definition 1.7 (Topology): Let $\Omega \neq \emptyset$ be an arbitrary set. A class of sets $\tau \subset 2\Omega$ is called a topology on Ω if it has the following three properties: (i) $\emptyset, \Omega \in \tau$. (ii) $A \cap B \in \tau$ for any $A, B \in \tau$. (iii) $(\cup_{A \in F} A) \in \tau$ for any $F \subset \tau$. The pair (Ω, τ) is called a **topological space**. The sets $A \in \tau$ are called open, and the sets $A \subset \Omega$ with $A^c \in \tau$ are called closed.

Definition 1.8 (Borel σ -algebra): Let (Ω, τ) be a topological space. The σ -algebra $B(\Omega) := B(\Omega, \tau) := \sigma(\tau)$ that is generated by the open sets is called the Borel σ -algebra on Ω . The elements $A \in B(\Omega, \tau)$ are called Borel sets or Borel measurable sets.

Definition 1.8 (Trace of a class of sets): Let $A \subset 2\Omega$ be an arbitrary class of subsets of Ω and let $A \in 2\Omega \setminus \{\emptyset\}$. The class $A|_A := \{A \cap B : B \in A\} \subset 2A$ is called the trace of A on A or the restriction of A to A .

Set Functions

Set Functions (cont)

σ -finite if there exists a sequence of sets $\Omega_1, \Omega_2, \dots \in A$ such that $\Omega = \cup_{n=1}^{\infty} \Omega_n$ and such that $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$.

Examples

Dirac measure Let $\omega \in \Omega$ and $\delta_\omega(A) = 1_A(\omega)$. Then δ_ω is a probability measure on any σ -algebra $A \subset 2\Omega$. δ_ω is called the

uniform distribution Let Ω be a finite nonempty set. By $\mu(A) := \#A / \#\Omega$ for $A \subset \Omega$, we define a probability measure on $A = 2\Omega$

Let $A \subset 2\Omega$ and let $\mu : A \rightarrow [0, \infty]$ be a set function. We say that μ is

monoton $\mu(A) \leq \mu(B)$ for any two sets $A, B \in A$ with $A \subset B$,

additiv if $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ for any choice of finitely many mutually disjoint sets $A_1, \dots, A_n \in A$ with $\cup_{i=1}^n A_i \in A$,

σ -additive if $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for any choice of countably many mutually disjoint sets $A_1, A_2, \dots \in A$ with $\cup_{i=1}^{\infty} A_i \in A$,

subadditive if for any choice of finitely many sets $A, A_1, \dots, A_n \in A$ with $A \subset \cup_{i=1}^n A_i$, we have $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$, and

σ -subadditive if for any choice of countably many sets $A, A_1, A_2, \dots \in A$ with $A \subset \cup_{i=1}^{\infty} A_i$, we have $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Let A be a semiring and let $\mu : A \rightarrow [0, \infty]$ be a set function with $\mu(\emptyset) = 0$. μ is called a

content if μ is additive,

premeasure if μ is σ -additive,

measure if μ is a premeasure and A is a σ -algebra

probability measure if μ is a measure and $\mu(\Omega) = 1$

Let A be a semiring. A content μ on A is called

finite if $\mu(A) < \infty$ for every $A \in A$



By **Julia22**
cheatography.com/julia22/

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